Comment(s) on "Control barrier functions for stochastic systems" [Automatica 130 (2021) 109688]

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Abstract

It is shown that Theorem 3 of Clark (2021) is flawed. The proof of Theorem 2 in Clark (2021) relies on the same proof technique and is thus also flawed.

Key words: Safe control, Stochastic control, Stochastic differential equations.

Below, we illustrate that the proof techniques used to prove Theorem 2 and Theorem 3 in Clark (2021) are both flawed. We show via a simple counterexample that Theorem 3 is false, and show why the proof in Clark (2021) is invalid.

Note that the incorrect result of Theorem 3 is also used to prove Theorem 7 in Clark (2021), which makes the latter result questionable. The same result has also been used in many other works e.g., (Pereira et al., 2021; Song et al., 2022; Vahs and Tumova, 2023a,b; Enwerem and Baras, 2023), which sheds doubt on the validity of many of the results in these papers.

1 Counterexample

We now provide a simple counterexample to illustrate that Theorem 3 of Clark (2021), which proves that a zero-CBF (ZCBF), defined in Definition 6 of Clark (2021), certifies that a system is safe with probability one, is incorrect.

Example 1 (Uncontrolled Brownian Motion)**.** Consider the case of (uncontrolled) one-dimensional Brownian Motion by taking

$$f(x_t) = 0, \quad g(x_t) = 0, \quad \sigma(x_t) = 1, \tag{1}$$

for the one-dimensional state $x_t \in \mathbb{R}$ and control $u_t \in \mathbb{R}$. The functions $f : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$, and $\sigma : \mathbb{R} \to \mathbb{R}$ are all locally Lipschitz continuous functions. This reduces the stochastic differential equation (SDE) (5) in Clark (2021) to $\mathrm{d}x_t = \mathrm{d}W_t$, which has the solution $x_t = x_0 + W_t$ for W_t a zero-mean Brownian motion as in (5) of Clark (2021). We now construct the ZCBF $h : \mathbb{R} \to \mathbb{R}$ as $h(x) = x$, where the safe operating region $\mathcal{C}$ corresponds to the non-negative reals $\mathcal{C} = \{x : x \geq 0\} = \mathbb{R}_{\geq 0}$.

We now verify that h satisfies the condition (8) in Definition 6 of Clark (2021) and hence is a ZCBF. In this case, condition (8) of Clark (2021) reduces to the condition that for all x satisfying $h(x) > 0$, there exists a u such that

$$0 \geq -h(x), \tag{2}$$

which holds since $h(x) > 0 \implies h(x) \geq 0$. Hence, h is a valid ZCBF per Definition 6 of Clark (2021). Thus, per Theorem 3 of Clark (2021), one obtains that if $x_0 \in \mathcal{C}$, then

$$\Pr(x_t \in \mathcal{C},\ \forall t) = \Pr(x_0 + W_t \geq 0,\ \forall t) = 1. \tag{3}$$

However, this is not true, since $x_0 + W_t$ is normally distributed with mean x_0 and variance t, hence $\Pr(W_t \geq 0, \forall t) \neq 1$. This disproves Theorem 3.

2 Flaw of the Proof

The main flaw in the proof of Theorem 3 in Clark (2021) comes from an implicit assumption when applying mathematical induction. For any $\theta \in [0, h(x_0)]$ (which covers the original setting of $\theta = \min\{\frac{\delta\epsilon}{2t}, h(x_0)\}$ from Clark (2021)), define the sequence of stopping times η_i and ζ_i for $i = 0, 1, \dots$ as in the proof of Theorem 3 in Clark

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(2021). Next, define the random process U_t as in (10) of Clark (2021). The flaw then comes from the following statement from the proof of Theorem 3 in Clark (2019):

> *We will first prove by induction that* $h(x_t) \geq U_t$ *and* $U_t \leq \theta$.

While the intention is to prove that this property holds for all $t \geq 0$, it turns out it is not possible to do so. In particular, the mathematical induction is performed by showing that this property holds for all compact intervals $[\eta_i, \zeta_i]$ and $[\zeta_i, \eta_{i+1}]$. However, unless the union of these intervals covers $[0, \infty)$, i.e.,

$$\lim_{i\to\infty} \eta_i = \lim_{i\to\infty} \zeta_i = \infty, \tag{4}$$

the property need not hold for all $t \geq 0$, rendering the proof invalid. As we show next, this indeed is not the case for the example of (uncontrolled) Brownian motion in Example 1.

Consider Example 1, where $dx_t = dW_t$ and $h(x_t) = x_t = x_0 + W_t$. Since $x_{\zeta_0} \geq \theta$, we must have that $x_{\eta_1} = \theta$, and thus $\{\zeta_1 = \eta_1\} \in \mathcal{F}^+_{\eta_1}$. Thus, by Blumenthal's 0-1 law of (Le Gall, 2016, Theorem 2.13), $\Pr(\zeta_1 = \eta_1) \in \{0, 1\}$. For any $\tau > 0$, since $W_{\eta_1+\tau} > \theta \implies \zeta_1 \leq \eta_1 + \tau$ by definition of ζ_1,

$$\Pr(W_{\eta_1+\tau} < \theta) \leq \Pr(\zeta_1 \leq \eta_1 + \tau). \tag{5}$$

Since $x_{\eta_1} = \theta$ implies $W_{\eta_1} = \theta - x_0$, by symmetry of Brownian motion (Le Gall, 2016, p.63), $\Pr(W_{\eta_1+\tau} < \theta - x_0) = \Pr(W_{\eta_1+\tau} > \theta - x_0) = \frac{1}{2}$. Hence, (5) implies that $\Pr(\zeta_1 \leq \eta_1 + \tau) \geq \frac{1}{2}$. Taking $\tau \downarrow 0$, we get that $\Pr(\zeta_1 = \eta_1) \geq \frac{1}{2}$. Hence, $\zeta_1 = \eta_1$ almost surely. Repeating this argument then shows that the stopping times $\eta_i, i = 1, 2, \ldots$ and $\zeta_i, i = 1, 2, \ldots$ are all equal almost surely, i.e.,

$$\eta_1 = \zeta_1 = \eta_2 = \ldots, \quad \text{a.s.} \tag{6}$$

Finally, since W_t is a Brownian motion, both $\zeta_0 = \inf\{t : x_0 + W_t > \theta\}$ and $\eta_1 = \inf\{t : x_0 + W_t < \theta,\ t > \zeta_0\}$ are finite almost surely. Hence, $\lim_{i\to\infty} \zeta_i = \lim_{i\to\infty} \eta_i$ is finite almost surely, and the union of these intervals does not cover $[0, \infty)$ in the case of Example 1.

3 Examining Theorem 2

The proof of Theorem 2 in Clark (2021) makes use of the same mathematical induction argument. Hence, the validity of the proof hinges on whether it can be shown that the union of these intervals covers $[0, \infty)$ or not, which the proof in Clark (2021) does not show.

4 Conclusion

In this comment article, we have identified a flaw in the almost-sure safety guarantees of stochastic ZCBFs as constructed in Theorem 3 of Clark (2021) and have shown that the proofs for both Theorem 2 and Theorem 3 of Clark (2021) are invalid. The appeal of ZCBFs as a way to guarantee safety almost surely without the need for unbounded drift has attracted much attention. Unfortunately, in light of the incorrectness of the result in (Clark, 2021) on which the results in these works rely, the claims in these works are questionable.

Acknowledgements

We would like to thank Andrew Clark for helpful discussions.

References

Andrew Clark. Control barrier functions for complete and incomplete information stochastic systems. In *2019 American Control Conference (ACC)*, pages 2928–2935. IEEE, 2019.

Andrew Clark. Control barrier functions for stochastic systems. *Automatica*, 130:109688, 2021.

Clinton Enwerem and John S Baras. Safe collective control under noisy inputs and competing constraints via non-smooth barrier functions. *arXiv preprint arXiv:2311.03284*, 2023.

Jean-François Le Gall. *Brownian motion, martingales, and stochastic calculus.* Springer, 2016.

Marcus Pereira, Ziyi Wang, Ioannis Exarchos, and Evangelos Theodorou. Safe optimal control using stochastic barrier functions and deep forward-backward sdes. In *Conference on Robot Learning*, pages 1783–1801. PMLR, 2021.

Lin Song, Neng Wan, Aditya Gahlawat, Chuyuan Tao, Naira Hovakimyan, and Evangelos A Theodorou. Generalization of safe optimal control actions on networked multiagent systems. *IEEE Transactions on Control of Network Systems*, 10(1):491–502, 2022.

Matti Vahs and Jana Tumova. Non-smooth control barrier functions for stochastic dynamical systems. *arXiv preprint arXiv:2309.06494*, 2023a.

Matti Vahs and Jana Tumova. Risk-aware control for robots with non-gaussian belief spaces. *arXiv preprint arXiv:2309.12857*, 2023b.